A SCHUR-TYPE ADDITION THEOREM FOR PRIMES

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Abstract. Suppose that all primes are colored with k colors. Then there exist monochromatic primes p_1, p_2, p_3 such that $p_1 + p_2 = p_3 + 1$.

1. Introduction

In [4], Green and Tao proved a celebrated result that the primes contain arbitrarily long non-trivial arithmetic progressions. In fact, they proved a Szemerédi-type [8] result for primes:

If A is a set of primes with positive relative upper density, then A contains arbitrarily long arithmetic progressions.

Thus if all primes are colored with k colors, then there exist arbitrarily long monochromatic arithmetic progressions. This is a van der Waerden-type [9] theorem for primes. (The well-known van der Waerden theorem states that for any k-coloring of all positive integers, there exist arbitrarily long monochromatic arithmetic progressions.)

On the other hand, Schur's theorem [7] is another famous result in the Ramsey theory for integers. Schur's theorem asserts that for any k-coloring of all positive integers, there exist monochromatic x, y, z such that $x + y = z$. In this paper, we shall prove a Schur-type theorem for primes.

Theorem 1.1. *Suppose that all primes are arbitrarily colored with k colors. Then there exist monochromatic primes p_1, p_2, p_3 such that $p_1 + p_2 = p_3 + 1$.*

Furthermore, motivated by the Green-Tao theorem and Theorem 1.1, we propose the following conjecture:

Conjecture 1.1. *Suppose that all primes are colored with k colors. Then for arbitrary $l \geq 3$, there exist monochromatic primes $p_0, p_1, p_2, \ldots, p_l$ such that $p_1, \ldots, p_l$ form an arithmetic progression with the difference $p_0 - 1$.*

Theorem 1.1 will be proved in the next section. And our proof uses a variant of Green's method [3] in his proof of Roth's theorem for primes.

2000 *Mathematics Subject Classification.* Primary 11P32; Secondary 05D10, 11B75.

This work was supported by the National Natural Science Foundation of China (Grant No. 10771135).

2. Proof of Theorem 1.1

Lemma 2.1. *Suppose that the set $\{1, 2, \dots, n\}$ is split into $A_1 \cup A_2 \cup \cdots \cup A_k$. Then there exists a constant $C_1(k) > 0$ such that*

$$\sum_{1 \le i \le k} |\{(x, y, z) : x, y, z \in A_i, x + y = z\}| \ge C_1(k) n^2$$

if n is sufficiently large.

This result is not new. In fact, Robertson and Zeilberger [5], Schoen [6] had showed that if the integers from 1 to n are colored with two colors, then there exist at least $(1/22 - \epsilon)n^2$ monochromatic Schur triples $\{x, y, x + y\}$. Furthermore, Robertson and Zeilberger [5] also claimed that for any k-coloring of $\{1, \dots, n\}$, the number of monochromatic Schur triples is greater than

$$(\frac{1}{2^{2k-3}11} - \epsilon)n^2.$$

However, for the sake of completeness, here we give a proof of Lemma 2.1. Suppose that $1, 2, \dots, n$ are colored with k colors. Let G be a complete graph with the vertex set $V = \{v_0, v_1, \dots, v_n\}$. Then we k-color all edges of G by giving the edge $v_s v_t$ the color of $t - s$ for every $0 \le s < t \le n$. Clearly for $0 \le r < s < t \le n$, three vertices v_r, v_s, v_t form a monochromatic triangle if and only if $\{s-r, t-s, t-r\}$ is a monochromatic Schur triple. And it is easy to see that one monochromatic Schur triple is corresponding to at most n monochromatic triangles. Hence Lemma 2.1 immediately follows from the next lemma:

Lemma 2.2. *Let G be a complete graph with n vertices. If all edges of G are colored with k colors, then there exist at least $C_1'(k)n^3$ monochromatic triangles provided that n is sufficiently large, where $C_1'(k) > 0$ is a constant only depending on k.*

Proof. Since G is a complete graph, G contains $\binom{n}{3}$ triangles. We use induction on k. There is nothing to do when $k = 1$. Assume that $k \ge 2$ and our assertion holds for any smaller value of k. Suppose that the vertex set V of G is $\{v_1, \dots, v_n\}$. Then for every $1 \le s \le n$, by the pigeonhole principle, there exist vertices $v_{t_{s,1}}, \dots, v_{t_{s,\lceil n/k \rceil}}$ and $1 \le c_s \le k$ such that the edge $v_s v_{t_{s,1}}, \dots, v_s v_{t_{s,\lceil n/k \rceil}}$ are colored with the c_s-th color, where $\lceil x \rceil$ denotes the smallest integer not less than x. Let us consider the $\binom{\lceil n/k \rceil}{2}$ edges between $v_{t_{s,1}}, \dots, v_{t_{s,\lceil n/k \rceil}}$. Suppose that at most $(C_1'(k-1)/2k^3)n^2$ of these edges are colored with the c_s-th color. Then by the induction hypothesis on $k - 1$, the remainder edges form at least

$$C_1'(k-1)(n/k)^3 - \frac{C_1'(k-1)}{2k^3} n^3 = \frac{C_1'(k-1)}{2k^3} n^3$$

monochromatic triangles, since one edge belongs to at most n triangles.

Then we may assume that for each $1 \le s \le n$, there exist at least $(C_1'(k-1)/2k^3)n^2$ edges between $v_{t_{s,1}}, \ldots, v_{t_{s,\lceil n/k\rceil}}$ are colored with the c_s-th color. Thus we get at least $(C_1'(k-1)/2k^3)n^2$ monochromatic triangles containing the vertex v_s. And there are totally at least

$$\frac{C_1'(k-1)}{6k^3}n^3$$

monochromatic triangles, by noting that every triangles are counted three times.
$\square$

Corollary 2.1. *Let A be a subset of $\{1, 2, \ldots, n\}$ with $|A| \ge (1 - C_1(k)/6)n$. Suppose that A is split into $A_1 \cup A_2 \cup \cdots \cup A_k$. Then*

$$\sum_{1 \le i \le k} |\{(x, y, z) : x, y, z \in A_i, x + y = z\}| \ge \frac{C_1(k)}{2}n^2$$

provided that n is sufficiently large.

Proof. Let $\bar{A} = \{1, \ldots, n\} \setminus A$. Then

$$\begin{aligned}
&|\{(x, y, z) : x, y, z \in A_1 \cup \bar{A}, x + y = z\}| \\
\le&|\{(x, y, z) : x, y, z \in A_1, x + y = z\}| \\
&+ |\{(x, y, z) : \text{one of } x, y, z \text{ lies in } \bar{A}, x + y = z\}| \\
\le&|\{(x, y, z) : x, y, z \in A_1, x + y = z\}| + 3|\bar{A}|n.
\end{aligned}$$

Hence by Lemma 2.1 we have

$$\begin{aligned}
&\sum_{1 \le i \le k} |\{(x, y, z) : x, y, z \in A_i, x + y = z\}| \\
\ge&|\{(x, y, z) : x, y, z \in A_1 \cup \bar{A}, x + y = z\}| - 3|\bar{A}|n \\
+&\sum_{2 \le i \le k} |\{(x, y, z) : x, y, z \in A_i, x + y = z\}| \\
\ge&\frac{C_1(k)}{2}n^2.
\end{aligned}$$

$\square$

Let $\mathcal{P}$ denote the set of all primes. Assume that $\mathcal{P} = P_1 \cup P_2 \cup \cdots \cup P_k$, where $P_i \cap P_j = \emptyset$ for $1 \le i < j \le k$. Let $w = w(n)$ be a function tending sufficiently slowly to infinity with n (e.g., we may choose $w(n) = \lfloor \frac{1}{4} \log\log n \rfloor$), and let

$$W = \prod_{\substack{p \in \mathcal{P} \\ p \leqslant w(n)}} p.$$

Clearly we have $W \leqslant \log n$ for sufficiently large n. Let

$$\kappa = \frac{C_1(k)}{10000k}.$$

In view of the well-known Siegel-Walfisz theorem, we may assume that n is sufficiently large so that

$$\sum_{\substack{x \in \mathcal{P} \cap [1,n] \\ x \equiv 1 \pmod{W}}} \log x \geq (1-\kappa)\frac{n}{\phi(W)},$$

where ϕ is the Euler totient function. Let $M = n/W$ and N be a prime in the interval $[(2+\kappa)M, (2+2\kappa)M]$. (Thanks to the prime number theorem, such prime N always exists whenever M is sufficiently large.) Define

$$\lambda_{b,W,N}(x) = \begin{cases} \phi(W)\log(Wx+b)/WN & \text{if } x \leq N \text{ and } Wx+b \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$A_0 = \{1 \leq x \leq M : Wx+1 \in \mathcal{P}\}$$

and

$$A_i = \{1 \leq x \leq M : Wx+1 \in P_i\}$$

for $1 \leq i \leq k$. Define

$$a_i(x) = \mathbf{1}_{A_i}(x)\lambda_{1,W,N}(x)$$

for $0 \leq i \leq k$, where we set $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 otherwise. Clearly we have $a_0 = a_1 + \cdots + a_k$ and

$$\sum_x a_0(x) = \sum_{1 \leq x \leq M} \lambda_{1,W,N}(x) \geq (1-\kappa)\frac{M}{N} \geq \frac{1}{2} - 3\kappa.$$

Below we consider $A_0, A_1, \ldots, A_k$ as the subsets of $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. Since $M < N/2$, if there exist $x, y, z \in A_i$ such that $x + y = z$ in $\mathbb{Z}_N$, then we have $p_1 + p_2 = p_3 + 1$ in $\mathbb{Z}$, where $p_1 = Wx + 1 \in P_i, p_2 = Wy + 1 \in P_i, p_3 = Wz + 1 \in P_i$. For a complex-valued function f over $\mathbb{Z}_N$, define $\tilde{f}$ by

$$\tilde{f}(r) = \sum_{x \in \mathbb{Z}_N} f(x)e(-xr/N),$$

where $e(x) = e^{2\pi\sqrt{-1}x}$. And for two functions f, g, define

$$(f * g)(x) = \sum_{y \in \mathbb{Z}_N} f(y)g(x-y).$$

It is easy to check that $(f * g)\tilde{} = \tilde{f}\tilde{g}$. Let $0 < \delta, \epsilon < 1/2$ be two sufficiently small real numbers which will be chosen later. Let

$$R = \{r \in \mathbb{Z}_N : \max_{1 \leq i \leq k} |\tilde{a}_i(r)| \geqslant \delta\}.$$

and

$$B = \{x \in \mathbb{Z}_N : x \in [-\kappa N, \kappa N],\ \|xr/N\| \leqslant 2\epsilon \text{ for all } r \in R\},$$

where $\|x\| = \min_{z\in\mathbb{Z}} |x - z|$. Here our definition of B is slightly different from Green's one in [3, Page 1629]. As we shall see later, this modification is the key of our proof.

Lemma 2.3.

$$|B| \geq \epsilon^{|R|}\kappa N.$$

Proof. Assume that $R = \{r_1, r_2, \ldots, r_m\}$. Let d be the greatest integer not exceeding $1/\epsilon$. Clearly we have $1/d \leq 2\epsilon$ since $\epsilon < 1/2$. Let

$$G_{t_1,\ldots,t_m} = \{-\kappa N/2 \leq x \leq \kappa N/2 : t_j/d \leq \{xr_j/N\} < (t_j + 1)/d \text{ for } 1 \leq j \leq m\},$$

where $\{\alpha\}$ denotes the fractional part of α. Clearly

$$\sum_{0\leq t_1,\ldots,t_m\leq d-1} |G_{t_1,\ldots,t_m}| = \kappa N.$$

Hence there exists a term of $(t_1, \ldots, t_m)$ such that

$$|G_{t_1,\ldots,t_m}| \geq d^{-m}\kappa N \geq \epsilon^m \kappa N.$$

For any given $x_0 \in G_{t_1,\ldots,t_m}$, when $x \in G_{t_1,\ldots,t_m}$, we have $x - x_0 \in [-\kappa N, \kappa N]$ and

$$\|(x - x_0)r_j/N\| \leq |\{xr_j/N\} - \{x_0 r_j/N\}| \leq 1/d \leq 2\epsilon$$

for $1 \leq j \leq m$. So $G_{t_1,\ldots,t_m} \subseteq x_0 + B$. This completes the proof. □

Lemma 2.4.

$$\sup_{r\neq 0} |\tilde{\lambda}_{b,W,N}(r)| \leq 2 \log\log w/w$$

provided that w is sufficiently large.

Proof. This is Lemma 6.2 of [3]. □

Let $\beta = \mathbf{1}_B/|B|$ and $a_i' = a_i * \beta * \beta$ for $0 \leq i \leq k$.

Lemma 2.5. *Suppose that $\epsilon^{|R|} \geq \kappa^{-2} \log\log w/w$. Then we have*

$$\sup_{x\in\mathbb{Z}_N} a_0'(x) \leq \frac{1 + 3\kappa}{N}.$$

Proof. We have

$$\begin{aligned}
a_0'(x) =& a_0 * \beta * \beta(x)\\
\le& \lambda_{1,W,N} * \beta * \beta(x)\\
=& N^{-1} \sum_{r \in \mathbb{Z}_N} \tilde{\lambda}_{1,W,N}(r)\tilde{\beta}(r)^2 e(xr/N)\\
\le& N^{-1}\tilde{\lambda}_{1,W,N}(0)\tilde{\beta}(0)^2 + N^{-1}\sup_{r\neq 0}|\tilde{\lambda}_{1,W,N}(r)| \sum_{r\in\mathbb{Z}_N} |\tilde{\beta}(r)|^2\\
=& N^{-1}\tilde{\lambda}_{1,W,N}(0) + \sup_{r\neq 0}|\tilde{\lambda}_{1,W,N}(r)| \sum_{r\in\mathbb{Z}_N} |\beta(r)|^2\\
\le& \frac{1+\kappa}{N} + \frac{2\log\log w}{w|B|},
\end{aligned}$$

where Lemma 2.4 is applied in the last step. Thus Lemma 2.5 immediately follows from Lemma 2.3. □

Lemma 2.6 (Bourgain [1, 2], Green [3])**.** *Let $\rho > 2$. For any function $f : \mathbb{Z}_N \to \mathbb{C}$,*

$$\sum_{r\in\mathbb{Z}_N} |(f\lambda_{b,W,N})\tilde{}(r)|^\rho \le C_2(\rho)\bigg(\sum_{x=1}^{N} |f(x)|^2 \lambda_{b,W,N}(x)\bigg)^{\frac{\rho}{2}}$$

where $C_2(\rho)$ is a constant only depending on ρ.

Proof. This is an immediate consequence of Theorem 2.1 and Lemma 6.5 of [3]. □

By Lemma 2.6, we have

$$\sum_{r\in\mathbb{Z}_N} |\tilde{a}_i(r)|^\rho \le C_2(\rho)$$

for $\rho > 2$ and $1 \le i \le k$. In particular,

$$\sum_{r\in R} \delta^3 \le \sum_{r\in\mathbb{Z}_N}\left(\sum_{i=1}^{k} |\tilde{a}_i(r)|^3\right) \le C_2(3)k,$$

which implies that $|R| \le C_2(3)\delta^{-3}k$.

Lemma 2.7. *For each $r \in R$,*

$$|1 - \tilde{\beta}(r)^4\tilde{\beta}(-r)^2| \le 384\epsilon^2.$$

Proof. By the definition of B, we have

$$|1-\tilde{\beta}(r)| = \frac{1}{|B|}\bigg|\sum_{x\in B}(1 - e(-xr/N))\bigg| \le 4\pi \sup_{x\in B}\|xr/N\|^2 \le 64\epsilon^2.$$

So

$$\begin{aligned}|1-\tilde{\beta}(r)^4\tilde{\beta}(-r)^2| =& \Big|\sum_{j=0}^{3}\tilde{\beta}(r)^j(1-\tilde{\beta}(r))+\tilde{\beta}(r)^4\sum_{j=0}^{1}\tilde{\beta}(-r)^j(1-\tilde{\beta}(-r))\Big|\\ \le& 384\epsilon^2.\end{aligned}$$

by noting that $|\tilde{\beta}(r)| \le \tilde{\beta}(0) = 1$. □

Lemma 2.8. *For* $1 \le i \le k$,

$$\Big|\sum_{i=1}^{k}\sum_{\substack{x,y,z\in\mathbb{Z}_N\\ x+y=z}} a_i(x)a_i(y)a_i(z) - \sum_{i=1}^{k}\sum_{\substack{x,y,z\in\mathbb{Z}_N\\ x+y=z}} a_i'(x)a_i'(y)a_i'(z)\Big| \le \frac{C_3k^2}{N}(\epsilon^2\delta^{-3}+\delta^{\frac{1}{3}}),$$

where C_3 *is an absolute constant.*

Proof. Clearly

$$\sum_{\substack{x,y,z\in\mathbb{Z}_N\\ x+y=z}} f_1(x)f_2(y)f_3(z) = N^{-1}\sum_{r\in\mathbb{Z}_N}\tilde{f}_1(r)\tilde{f}_2(r)\tilde{f}_3(-r).$$

Hence

$$\begin{aligned}&\sum_{i=1}^{k}\sum_{\substack{x,y,z\in\mathbb{Z}_N\\ x+y=z}} a_i(x)a_i(y)a_i(z) - \sum_{i=1}^{k}\sum_{\substack{x,y,z\in\mathbb{Z}_N\\ x+y=z}} a_i'(x)a_i'(y)a_i'(z)\\ =& N^{-1}\sum_{i=1}^{k}\sum_{r\in\mathbb{Z}_N}\tilde{a}_i(r)^2\tilde{a}_i(-r)(1-\tilde{\beta}(r)^4\tilde{\beta}(-r)^2).\end{aligned}$$

By Lemma 2.7,

$$\begin{aligned}&\Big|\sum_{i=1}^{k}\sum_{r\in R}\tilde{a}_i(r)^2\tilde{a}_i(-r)(1-\tilde{\beta}(r)^4\tilde{\beta}(-r)^2)\Big|\\ \le& 384\epsilon^2k|R|\sup_{r}\max_{1\le i\le k}|\tilde{a}_i(r)|^3\\ \le& 384C_2(3)\epsilon^2\delta^{-3}k^2,\end{aligned}$$

since $|\tilde{a}_i(r)| \leq \tilde{a}_i(0) \leq 1$. On the other hand, by the Hölder inequality, we have

$$\begin{aligned}
&\Big|\sum_{i=1}^{k}\sum_{r\notin R}\tilde{a}_i(r)^2\tilde{a}_i(-r)(1-\tilde{\beta}(r)^4\tilde{\beta}(-r)^2)\Big| \\
\leq& 2\sum_{i=1}^{k}\sum_{r\notin R}|\tilde{a}_i(r)|^2|\tilde{a}_i(-r)| \\
\leq& 2\sup_{r\notin R}\max_{1\leq i\leq k}|\tilde{a}_i(r)|^{\frac{1}{3}}\Big(\sum_{i=1}^{k}\sum_{r}|\tilde{a}_i(r)|^{\frac{5}{2}}\Big)^{\frac{2}{3}}\Big(\sum_{i=1}^{k}\sum_{r}|\tilde{a}_i(-r)|^{3}\Big)^{\frac{1}{3}} \\
\leq& 2C_2(5/2)^{\frac{2}{3}}C_2(3)^{\frac{1}{3}}\delta^{\frac{1}{3}}k.
\end{aligned}$$

We choose $C_3 = 384C_2(3) + 2C_2(5/2)^{\frac{2}{3}}C_2(3)^{\frac{1}{3}}$, then the Lemma follows. $\square$

Define

$$X = \{x \in \mathbb{Z}_N : a_0'(x) \geq \frac{\kappa}{N}\}.$$

Then by Lemma 2.5, we have

$$\frac{1+3\kappa}{N}|X| + \frac{\kappa}{N}(N-|X|) \geq \sum_{x\in\mathbb{Z}_N} a_0'(x) = \sum_{x\in\mathbb{Z}_N} a_0(x) \geq \frac{1}{2} - 3\kappa.$$

It follows that

$$|X| \geq \big(\frac{1}{2} - 6\kappa\big)N.$$

Notice that $\mathrm{supp}(a_i) \subseteq [1, M]$ and $\mathrm{supp}(\beta) \subseteq [-\kappa N, \kappa N]$, where

$$\mathrm{supp}(f) = \{x \in \mathbb{Z}_N : f(x) \neq 0\}.$$

Hence

$$\mathrm{supp}(a_i') = \mathrm{supp}(a_i * \beta * \beta) \subseteq [-2\kappa N, M + 2\kappa N]$$

for $0 \leq i \leq k$. Thus we have

$$X \subseteq \mathrm{supp}(a_0') \subseteq [-2\kappa N, M + 2\kappa N].$$

Let $A_0' = X \cap [1, M]$. Then

$$|A_0'| \geq |X| - 4\kappa N \geq (1 - 20\kappa)M,$$

by recalling that $(2+\kappa)M \leq N \leq (2+2\kappa)M$. Since

$$a_0' = a_0 * \beta * \beta = (a_1 + \cdots + a_k) * \beta * \beta = a_1' + \cdots + a_k',$$

we have

$$\max_{1\leq i\leq k} a_i'(x) \geq \frac{\kappa}{kN}$$

for each $x \in A_0'$. Let

$$X_i = \{x \in A_0' : a_i'(x) = \max_{1\leq i\leq k} a_i'(x)\}.$$

Clearly $A_0' = X_1 \cup \cdots \cup X_k$. Let $A_1' = X_1$ and

$$A_i' = X_i \setminus \left(\bigcup_{j=1}^{i-1} X_j \right)$$

for $2 \le i \le k$. Then $A_1', \ldots, A_k'$ form a partition of A_0'. Furthermore, for $1 \le i \le k$ and each $x \in A_i'$, we have

$$a_i'(x) \ge \frac{\kappa}{kN}.$$

Thus by Corollary 2.1 and Lemma 2.8

$$\begin{aligned}
\sum_{i=1}^{k} \sum_{\substack{x,y,z \in \mathbb{Z}_N \\ x+y=z}} a_i(x)a_i(y)a_i(z) \ge & \sum_{i=1}^{k} \sum_{\substack{x,y,z \in \mathbb{Z}_N \\ x+y=z}} a_i'(x)a_i'(y)a_i'(z) - \frac{C_3 k^2}{N}(\epsilon^2 \delta^{-3} + \delta^{\frac{1}{3}}) \\
\ge & \sum_{i=1}^{k} \sum_{\substack{x,y,z \in A_i' \\ x+y=z}} \left(\frac{\kappa}{kN} \right)^3 - \frac{C_3 k^2}{N}(\epsilon^2 \delta^{-3} + \delta^{\frac{1}{3}}) \\
\ge & \left(\frac{\kappa}{kN} \right)^3 \frac{C_1(k) M^2}{2} - \frac{C_3 k^2}{N}(\epsilon^2 \delta^{-3} + \delta^{\frac{1}{3}})
\end{aligned}$$

Finally, we may choose sufficiently small δ and ϵ with

$$\epsilon^{-C_2(3)\delta^{-3} k} \ge \kappa^{-2} \log\log w / w$$

such that

$$\epsilon^2 \delta^{-3} + \delta^{\frac{1}{3}} \le \frac{C_1(k) \kappa^3}{24 C_3 k^5},$$

whenever N is sufficiently large. Thus

$$\sum_{i=1}^{k} \sum_{\substack{x,y,z \in \mathbb{Z}_N \\ x+y=z}} a_i(x)a_i(y)a_i(z) \ge \frac{C_1(k)\kappa^3 M^2}{2k^3 N^3} - \frac{C_1(k)\kappa^3}{24k^3 N} \ge \frac{C_1(k)\kappa^3}{12k^3 N} - \frac{C_1(k)\kappa^3}{24k^3 N} > 0.$$

This completes the proof. □

Remark. Notice that

$$\sum_{i=1}^{k} \sum_{\substack{x,z \in \mathbb{Z}_N \\ 2x=z}} a_i(x)^2 a_i(z) = O\left(\frac{k\phi(W)^3 \log(WN+1)^3}{W^3 N^2} \right) = o(N^{-1}).$$

Hence in fact there exist three distinct monochromatic primes p_1, p_2, p_3 satisfying $p_1 + p_2 = p_3 + 1$.

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